Calculating Far-Field Radiation Based on FEKO Spherical Wave Coefficients

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I. INTRODUCTION

Numerical electromagnetic simulation packages, such as FEKO (www.feko.info), most typically provide far-field data at constant \( \Delta \theta, \Delta \phi \) steps. This works fine for antenna applications, but is inconvenient in Radio Astronomy as celestial sources do not generally follow constant \( \theta \) or \( \phi \) trajectories. However, an option to calculate Spherical Wave Expansion (SWE) coefficients is provided in FEKO. This allows calculation of continuous (near and far) fields at radii larger than that of the sphere containing the sources [1], [2]. Radio astronomy deals with far-field radiation, and hence, a far-field expression is sufficient for our purpose.

II. FAR-FIELD EXPRESSION USING FEKO’S SWE

We follow FEKO’s SWE convention as described in [1]. In the far-field \( (r \to \infty) \), the electric field can be expressed as:

\[
\hat{E}^H (r, \theta, \phi) = \beta \sqrt{\frac{Z_0}{2 \pi r}} e^{-j \beta r} \left[ \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{e^{j m \phi} C_{mn}}{\sqrt{n(n+1)}} \left( \frac{m}{|m|} \right)^m (e^{\theta \hat{\theta}} + e^{\phi \hat{\phi}}) \right]
\]

(1)

where \( \beta \) is the wavenumber and \( Z_0 \) is the intrinsic impedance of free space and

\[
epsilon_{mn}^\theta = Q_{1mn} j^{n+1} \frac{j m}{\sin \theta} P_n^{|m|}(\cos \theta) + Q_{2mn} j^n \frac{dP_n^{|m|}(\cos \theta)}{d\theta}
\]

(2)

\[
epsilon_{mn}^\phi = Q_{2mn} j^n P_n^{|m|}(\cos \theta) - Q_{1mn} j^{n+1} \frac{dP_n^{|m|}(\cos \theta)}{d\theta}
\]

(3)

\( Q_{snm} \) are the coefficients given by FEKO where \( s = 1 \) and \( s = 2 \) refer to TE and TM modes, respectively. Similar expressions, though with slightly different conventions, may be found in [3], [4]. Also,

\[
C_{mn} = \sqrt{\frac{2n+1 (n-|m|)!}{2 (n+|m|)!}}
\]

(4)

is the normalization factor for the associated Legendre function of order \( n \) and rank \( |m| \), \( P_n^{|m|}(\cos \theta) \) [5], [6].

A. Dealing with \( P_n^{|m|}(\cos \theta)/\sin \theta \)

The factor \( P_n^{|m|}(\cos \theta)/\sin \theta \) gives an appearance of singularity for \( \theta \to 0, \pi \) which requires special treatment. Note that \( \theta = 0 \) is in the direction of the zenith in LFAA; it is important that we get this right. We can use a solution to the associated Legendre equation given by [6]:

\[
P_n^{|m|}(u) = (-1)^{|m|} (1 - u^2)^{|m|/2} \frac{d^{|m|} P_n(u)}{du^{|m|}}
\]

(5)

where \( u = \cos \theta \). It follows that

\[
\frac{P_n^{|m|}(\cos \theta)}{\sin \theta} = \frac{P_n^{|m|}(u)}{(1 - u^2)^{1/2}} = (-1)^{|m|} (1 - u^2)^{|m|-1/2} \frac{d^{|m|} P_n(u)}{du^{|m|}}
\]

(6)

There are three cases to consider:

1) \( m = 0 \): In (2) and (3) (and as it turns out, in all cases encountered here), the \( P_n^{|m|}(\cos \theta)/\sin \theta \) factor is multiplied by such that

\[
mP_n(\cos \theta)/\sin \theta \quad \theta \to 0, \pi \quad 0
\]

(7)
2) \(|m| = 1|\): Setting \(|m| = 1|\) in (6), we obtain

\[
\frac{P_n(\cos \theta)}{\sin \theta} = -\frac{dP_n(u)}{du}
\]  
(8)

From the definition of the Legendre polynomial \([6]\)

\[
P_n(u) = \sum_{l=0}^{L} \frac{(-1)^l(2n-2l)!}{2^n l!(n-l)!(n-2l)!} u^{n-2l}
\]

where \(L = n/2 \) (for \(n\) even) or \((n-1)/2 \) (for \(n\) odd). Therefore, we can write

\[
\frac{dP_n(u)}{du} = \sum_{l=0}^{L} \frac{(-1)^l(2n-2l)!}{2^n l!(n-l)!(n-2l)!} u^{n-2l-1}
\]

which allows us to obtain, for \(\theta \rightarrow 0, \pi\):

\[
\frac{dP_n(u)}{du} = \sum_{l=0}^{L} \frac{(-1)^l(2n-2l)!}{2^n l!(n-l)!(n-2l)!} (\pm 1)^{n-2l-1}
\]

(10)

3) \(|m| \geq 2|\): From (6), we obtain

\[
P_n^{[m]}(\cos \theta)/\sin \theta = (-1)^{|m|}(\sin \theta)^{|m|-1}\frac{d^{[m]}P_n(\cos \theta)}{d(\cos \theta)^{|m|}}
\]

(12)

Equation (9) suggests that \(P_n(u)\) is continuously differentiable for 
\(|u| \leq 1|\), hence

\[
P_n^{[m]}(\cos \theta)/\sin \theta \rightarrow 0
\]

(13)

Table I summarizes our discussion in this subsection. Note that the pre-multiplying factor, \(m\), is included.

| \(|m|\) | \(\lim_{\theta \rightarrow 0, \pi} \frac{mP_n^{[m]}(\cos \theta)}{\sin \theta}\) |
|---|---|
| 0 | 0 |
| 1 | \(-m\sum_{l=0}^{L=\text{floor}(n/2)} \frac{(-1)^l(2n-2l)!}{2^n l!(n-l)!(n-2l)!} (\cos(\theta = 0, \pi))^{n-2l-1}\) |
| \(\geq 2\) | 0 |

B. Dealing with \(dP_n^{[m]}(\cos \theta)/d\theta\)

We are interested in

\[
\frac{dP_n^{[m]}(\cos \theta)}{d\theta} = -\sin \theta \frac{P_n^{[m]}(\cos \theta)}{d(\cos \theta)}
\]

(14)

For the second factor in the right-hand-side (RHS) of (14), we can use a derivative formula given in [6]

\[
\frac{dP_n^{[m]}(u)}{du} = \frac{|m|u}{1-u^2} P_n^{[m]}(u) - \frac{P_n^{[m]+1}(u)}{(1-u^2)^{1/2}}
\]

(15)

With that substitution, we obtain

\[
\frac{dP_n^{[m]}(\cos \theta)}{d\theta} = \frac{|m|u}{\sqrt{1-u^2}} P_n^{[m]}(u) + P_n^{[m]+1}(u)
\]

\[
= \cos \theta \frac{|m|P_n^{[m]}(\cos \theta)}{\sin \theta} + P_n^{[m]+1}(\cos \theta)
\]

(16)
We again encounter \( P_n^{|m|}(\cos \theta)/\sin \theta \) factor in the RHS of (16) for which we can consult Table I for \( \theta \rightarrow 0, \pi \). The only exception is for \(|m| = 1 \) where the pre-multiplying factor is \(-|m| = -1\).

The discussion above allows us to re-write (2), (3) as

\[
e^n_m = j^n \left[ \frac{P_n^{|m|}(\cos \theta)}{\sin \theta} \left( m|Q_{2m} \cos \theta - mQ_{1m} \right) + Q_{2m}P_n^{|m|+1}(\cos \theta) \right]
\]

\[
e^\phi_m = j^{n+1} \left[ \frac{P_n^{|m|}(\cos \theta)}{\sin \theta} \left( mQ_{2m} - |m|Q_{1m} \cos \theta \right) - Q_{1m}P_n^{|m|+1}(\cos \theta) \right]
\]

III. Simple Examples

A. Single Hertzian dipole

1) \( +\hat{z} \)-directed Hertzian dipole with \( l \Delta l = 1 \) Am at origin: This is a single TM_{m=0,n=1} mode. From FEKO, \( Q_{201} = -93.7 [\sqrt{W}] \). We are left with

\[
\hat{E}^\|_{\theta, \phi}(r, \phi) = \beta \sqrt{\frac{Z_0}{2\pi}} e^{-j\beta r} \sqrt{\frac{3}{4}} e^\theta_{01} \hat{\theta}
\]

where

\[
e^\theta_{01} = jQ_{201}P_1^1(\cos \theta) = jQ_{201}(-\sin \theta) = j93.7 \sin \theta [\sqrt{W}]
\]

The \( \sin \theta \) radiation pattern and \( \hat{\theta} \) only polarization are expected. Neglecting the \( e^{-j\beta r}/r \) factor (implicitly assumed henceforth) and using \( Z_0 = 367.73 \Omega \), we obtain \( \hat{E}^\|(\pi/2, 0) = j628.3 \hat{\theta} \) which is identical to \( j628.3 \) given by FEKO.

2) \( +\hat{y} \)-directed Hertzian dipole with \( l \Delta l = 1 \) Am at origin: From FEKO, \( Q_{2, -1, 1} = Q_{211} = j66.25 [\sqrt{W}] \)

\[
\hat{E}^\|_{\theta, \phi}(r, \phi) = \beta \sqrt{\frac{Z_0}{2\pi}} e^{-j\beta r} \Sigma
\]

\[
\Sigma = \sqrt{\frac{3}{8}} \left( e^\theta_{11} \hat{\theta} + e^\phi_{11} \hat{\phi} \right) + e^{-j\phi} \sqrt{\frac{3}{8}} \left( e^\theta_{-11} \hat{\theta} + e^\phi_{-11} \hat{\phi} \right)
\]

where

\[
e^\theta_{-11} = -jQ_{2, -1, 1} \cos \theta \\
e^\theta_{11} = -jQ_{211} \cos \theta \\
e^\phi_{-11} = -Q_{2, -1, 1} \\
e^\phi_{11} = Q_{211}
\]

substituting the values for \( Q_{2, -1, 1} \) and \( Q_{211} \), we obtain

\[
\Sigma = -2j 66.25 \sqrt{\frac{3}{8}} \left( \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \right)
\]

Again, \( \hat{E}^\|(0, 0) = -j628.3 \hat{\phi} \) is identical to \(-j628.3 \) given by FEKO.

3) \( +\hat{x} \)-directed Hertzian dipole with \( l \Delta l = 1 \) Am at origin: From FEKO, \( Q_{2, -1, 1} = -Q_{211} = 66.25 [\sqrt{W}] \). Re-using (21) and (22), we obtain

\[
\Sigma = -2j 66.25 \sqrt{\frac{3}{8}} \left( \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \right)
\]

Note that the patterns expressed in (23) and (24) are consistent with the Jones matrix of crossed \( \hat{x} \) and \( \hat{y} \) Hertzian dipoles [7].

\(^1\text{using } \mu_0 = 4\pi \times 10^{-7} \text{ and } \epsilon_0 = 8.854 \times 10^{-12}. \text{ This more closely matches the value used in FEKO as opposed to 377 or 120\pi \Omega}\)
B. Array of Hertzian dipoles

Consider two Hertzian dipoles, +\( \hat{y} \)-directed at \((0,0,\lambda/20)\) and −\( \hat{y} \)-directed at \((0,0,-\lambda/20)\), each with \( I\Delta l = 1 \) Am. FEKO SWE coefficients for this problem are: \(-Q_{1,-1,1} = Q_{111} = 20.6; Q_{2,-1,2} = Q_{212} = 15.9; -Q_{1,-1,3} = Q_{113} = 0.089.\) We neglect \(-Q_{1,-1,3}, Q_{113}\) (very small) values for simplicity.

It can be shown that

\[
\Sigma \approx \frac{C_{11}}{\sqrt{2}} \left( e^{\theta} e^{-j\phi} - e^{\theta} e^{j\phi} \right) \hat{\theta} + \frac{C_{12}}{\sqrt{6}} \left( e^{\theta} e^{-j\phi} - e^{\theta} e^{j\phi} \right) \hat{\phi} + e^{\theta} e^{-j\phi} \hat{\phi} + e^{\theta} e^{j\phi} \hat{\theta}
\]

Substituting the coefficients, we obtain

\[
\Sigma \approx 50.4 \cos \theta \left( \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \right)
\]

This radiation pattern is proportional to \( \cos \theta \) times (23). The \( \cos \theta \) factor can be seen as the array factor of two closely spaced and oppositely signed point sources: \( \sin\left(\beta \lambda/20 \cos \theta\right) \approx \beta \lambda/20 \cos \theta.\) Here, \( \hat{E}_h(0,0) = 390 \) \( \hat{\phi} \) which is similar to 388.3 given by FEKO. This small difference seems to be due to the neglected \( Q_{1,-1,3}, Q_{113} \) factors.

IV. NUMERICAL IMPLEMENTATION AND EXAMPLES

A. Implementation

We find equations (17) and (18) in conjunction with (1) to be very convenient for numerical implementation. Two aspects are worth mentioning:

1) FEKO *.out file prints (“FAR FIELD MODAL COEFFICIENTS”) \( Q_{1mn} \) and \( Q_{2mn} \) alternately (as a column vector) with increasing order \( m = -n \) to \( n \) for every degree \( n.\) Once the \( Q_{1mn} \) and \( Q_{2mn} \) are separated into two column vectors, the FEKO \((m,n)\) format is convenient as it is compatible with \texttt{legendre(n,u)} function found in MATLAB.

2) \( P_n^{[m]}(\cos \theta)/\sin \theta \) and \( P_n^{[m+1]}(\cos \theta) \) are easily implemented using \texttt{legendre(n,u)}. We deal with apparent singularities in the former as per Tab. I.

3) Numerical calculation of the factorials in (11) appears to be unstable for \( N \sim 45.\) Consequently, we employ forward and backward differencing to approximate (8) numerically.

B. Example: closely spaced ±\( \hat{y} \) Hertzian dipoles

We return to the example in Sec. III-B, this time testing our numerical implementation. All FEKO SWE coefficients including \( Q_{1,-1,3} = Q_{113} \) are used. The analytical expression for this problem is:

\[
\hat{E}_h(\theta,\phi) = \frac{2I\Delta l}{4\pi} Z_0 \beta \sin(\frac{\pi}{\lambda} \frac{z}{\lambda} \cos \theta) \left( \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \right)
\]

where \( I\Delta l = 1 \) Am and \( z/\lambda = 1/20.\)

Fig. 1 reports comparison between analytical expression and numerical calculation based on spherical harmonics for the \((\theta,\phi)\) trajectory indicated. This trajectory is representative of of a celestial source, Hydra, as seen from the Murchison Radio-astronomy Observatory (MRO) in Western Australia. The difference between the two curves of less than 0.25% is very small.

C. Example: Antenna Array on Soil

The next example is a pseudo random array of 16 dual-polarized log-periodic antennas (referred to as AAVS0.5) on MRO soil [8], [9]. Fig. 2 depicts the simulation setup in FEKO. The array is pointed to Azimuth/Elevation of 0/75 degrees. Fig. 3 reports antenna gains at the nominal pointing direction taken from FEKO far-field data and computed via spherical harmonics over frequency. The two results are in excellent agreement with no more than \( \sim 0.5\% \) difference.

V. CONCLUSION

Spherical harmonics expansion is a convenient method for calculating continuous far-field radiation. This is especially useful in radio astronomy where celestial sources follow trajectories that continuously vary in \( \theta, \phi \) in the spherical coordinate system. We discussed an implementation based on FEKO generated spherical modal coefficients and found very good agreement with far-field values calculated by FEKO and analytical expression.

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Figure 1. Comparison between numerically calculated far-field based on spherical harmonics and analytical expression. The difference is less than 0.25%.

Figure 2. FEKO simulation of a pseudo random array of 16 dual-polarized log-periodic antennas distributed in an 8 m diameter circle.

REFERENCES

Figure 3. Antenna gains for Y (N-S) and X (E-W) polarization taken from FEKO far-field data and calculated from spherical wave expansion.